CONVERGENCE OF JACOBI-TYPE ALGORITHMS FOR SIMULTANEOUS APPROXIMATE DIAGONALIZATION OF MATRICES OR TENSORS

Konstantin Usevich

Université de Lorraine, CNRS,
CRAN
F-54000 Nancy, France
konstantin.usevich@univ-lorraine.fr

Jianze Li and Pierre Comon

Univ. Grenoble Alpes, CNRS
Grenoble INP, GIPSA-lab
F-38000 Grenoble, France
{firstname.lastname}@gipsa-lab.fr

ABSTRACT

Approximate orthogonal/unitary diagonalization of matrices and tensors is at the core of many source separation algorithm. We consider a family of Jacobi-type algorithms for approximate diagonalization (including the JADE and CoM algorithms). We report recent results on local and global convergence of these algorithms.

1. INTRODUCTION

Diagonalization of cumulant tensors or covariance matrices by orthogonal transformations are at the core of classic algorithms for blind source separation [1]. Several algorithms proposed in the literature perform approximate diagonalization by successive Jacobi rotations. The first were the Jacobi CoM (Contrast Maximization) algorithm for orthogonal diagonalization of 3rd and 4th-order real symmetric (cumulant) tensors [2, 3] and the JADE (Joint Approximate Diagonalization of Eigenmatrices) algorithm for simultaneous orthogonal/unitary (covariance) matrix diagonalization [4]. An algorithm for simultaneous orthogonal 3rd-order tensor diagonalization was proposed in [5], that can be applied to slices of 4th order cumulant tensors.

Jacobi-type algorithms are very attractive due to the fact that Jacobi rotations can be computed by rooting low-order polynomials, thus the iterations are very fast. Nevertheless, to our knowledge, the convergence of these methods was not proved, although it was often observed in practice.

In the recent paper [6] the real case was considered, and several modifications of the classic cyclic Jacobi algorithm. The first modification is the gradient-based order of Jacobi rotations proposed initially in [7] for best low multilinear rank approximation of 3rd-order symmetric tensors. The second modification proposed in [6] is an algorithm of proximal type. For both modifications local and global convergence properties were proved in [6]. In this note, we summarize the results of [6] for the gradient-based algorithm and discuss extension of [6] to the complex-valued case.

2. ORTHOGONAL DIAGONALIZATION

Let $\{\boldsymbol{\mathcal{A}}^{(\ell)} : 1 \leq \ell \leq m\} \subset \mathbb{R}^{n \times \cdots \times n}$ be a set of symmetric tensors. We wish to maximize

$$\boldsymbol{Q}_* = \arg \max_{\boldsymbol{Q} \in \mathcal{SO}_n} \sum_{\ell=1}^{m} \| \operatorname{diag}\{\boldsymbol{\mathcal{W}}^{(\ell)}\}\|^2, \qquad (1)$$

where $\boldsymbol{\mathcal{W}}^{(\ell)} = \boldsymbol{\mathcal{A}}^{(\ell)} \bullet_1 \boldsymbol{Q}^{\mathsf{T}} \cdots \bullet_d \boldsymbol{Q}^{\mathsf{T}}$ for $1 \leq \ell \leq m$, $\bullet_k$ is the k-mode product and $\mathcal{SO}_n$ is the special orthogonal group. Problem (1) has the following special cases:

- tensor diagonalization if $m = 1, d > 2$;
- simultaneous matrix diagonalization if $m > 1, d = 2$.

Let $\boldsymbol{G}^{(i,j,\theta)} \in \mathbb{R}^{n \times n}$ be the Givens rotation matrix

$$(\boldsymbol{G}^{(i,j,\theta)})_{k,l} = \begin{cases} 1, & k = l, k \notin \{i, j\}, \\ \cos\theta, & k = l, k \in \{i, j\}, \\ \sin\theta, & (k, l) = (j, i), \\ -\sin\theta, & (k, l) = (i, j), \\ 0, & \text{otherwise} \end{cases}$$

for $1 \leq k, l \leq n$. The classic Jacobi algorithm is based on the successive Jacobi rotations.

Algorithm 1 (Jacobi-C algorithm) Input: *Point* $\boldsymbol{Q}_0$. **Output:** *Sequence of iterations* $\{\boldsymbol{Q}_k\}_{k \geq 1}$.

- **For** $k = 1, 2, \ldots$ *[until ...] do*
- *Choose the pair* (i_k, j_k) *according to the cyclic pair selection rule:*

$$(1, 2) \to (1, 3) \to \cdots \to (n - 1, n) \to \cdots \qquad (2)$$

- *Compute the angle* θ_k^* *that maximizes the function*

$$h_k(\theta) \stackrel{\text{def}}{=} f(\boldsymbol{Q}_{k-1} \boldsymbol{G}^{(i_k, j_k, \theta)}). \qquad (3)$$

- *Set* $\boldsymbol{U}_k \stackrel{\text{def}}{=} \boldsymbol{G}^{(i_k, j_k, \theta_k^*)}$*, and update* $\boldsymbol{Q}_k = \boldsymbol{Q}_{k-1} \boldsymbol{U}_k$.
- **End for**

Special cases of algorithm 1 include the CoM algorithm [2, 3] and JADE [4] (in the real-valued case), and STOTD [5]; In all these cases the function (3) can be maximized by rooting a polynomial of order at most 4.

In every iteration the cost function increases, and due to compactness of $\mathcal{SO}_n$, the sequence of cost function values $f(\boldsymbol{Q}_1), f(\boldsymbol{Q}_2), \ldots$ is nondecreasing and has a limit point. Nevertheless, the following remarks should be made. First, the sequence $\boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots$ is not guaranteed to converge to a single point (which is desirable for identifiability in source separation), and potentially may evolve cyclicly. Second, even if a convergent subsequence is extracted, the subsequence is not guaranteed to converge to a stationary point. It is difficult to study these properties for the cyclic Jacobi algorithm, and we prove them for two modifications instead.

3. GRADIENT-BASED JACOBI ALGORITHM

The first modification was proposed in [7]. Following [7, Lemma 5.1], we define the projected gradient as

$$\mathrm{Proj}\nabla f(\boldsymbol{Q}) \stackrel{\mathsf{def}}{=} \boldsymbol{Q}\boldsymbol{\Lambda}(\boldsymbol{Q}), \tag{4}$$

$$\text{where} \quad \boldsymbol{\Lambda}(\boldsymbol{Q}) \stackrel{\mathsf{def}}{=} \frac{\boldsymbol{Q}^{\mathsf{T}}\nabla f(\boldsymbol{Q}) - (\nabla f(\boldsymbol{Q}))^{\mathsf{T}}\boldsymbol{Q}}{2} \tag{5}$$

and $\nabla f(\boldsymbol{Q})$ is the Euclidean gradient of f. The idea of [7], is to a choose pair (i, j) at each iteration that satisfies

$$|\langle \mathrm{Proj}\nabla f(\boldsymbol{Q}), d_{i,j}(\boldsymbol{Q})\rangle| \geq \varepsilon \| \mathrm{Proj}\nabla f(\boldsymbol{Q})\|, \tag{6}$$

where ε is a constant and $d_{i,j}(\boldsymbol{Q}) = \boldsymbol{Q}\frac{\partial \boldsymbol{G}^{(i,j,\theta)}}{\partial \theta}|_{\theta=0}$. Condition (6) ensures that the rotations are well-aligned with the gradient.

Algorithm 2 (Jacobi-G algorithm) Input: *A small positive* $0 < \varepsilon \leq \frac{2}{n}$*, a starting point* $\boldsymbol{Q}_0$.
Output: *Sequence of iterations* $\{\boldsymbol{Q}_k\}_{k\geq 1}$.

- **For** $k = 1, 2, \ldots$ *[until ....] do*
- *Choose* $(i, j) = (i_k, j_k)$ *satisfying* (6) *at* $\boldsymbol{Q} = \boldsymbol{Q}_{k-1}$.
- *Compute the angle* θ_k^* *that maximizes the function* $h_k(\theta)$ *defined in* (3).
- *Set* $\boldsymbol{U}_k \stackrel{\mathsf{def}}{=} \boldsymbol{G}^{(i_k, j_k, \theta_k^*)}$*, and update* $\boldsymbol{Q}_k = \boldsymbol{Q}_{k-1}\boldsymbol{U}_k$.
- **End for**

The following convergence properties can be proved.

Theorem 1 ([7, Theorem 5.4], [6, Theorem 3.3]) *Every accumulation point*[1] $\boldsymbol{Q}_*$ *of* $\{\boldsymbol{Q}_k\}_{k\geq 1}$ *produced by algorithm 2 is a stationary point of* f *(*i.e, $\mathrm{Proj}\nabla f(\boldsymbol{Q}_*) = 0$*).*

Theorem 2 ([6, Theorem 5.6]) *For* $d = 2, 3$ *and any* $\boldsymbol{Q}_0$*, Algorithm 2 converges to a stationary point of* f *in* $\mathcal{SO}_n$.

Theorem 2 was proved in [6] by employing the Lojasiewicz gradient inequality.

[1] i.e., the limit of every convergent subsequence.

4. DISCUSSION

In [6] the global convergence of two Jacobi-type algorithms (Algorithm 2 and an algorithm of proximal type) was proved for the real-valued case. In the complex-valued case, we are looking at a similar problem of joint diagonalization of a set of complex Hermitian matrices, as in the algorithm of [4]. The main difficulty is that the optimal Jacobi rotations are no longer univariate, and are similar to subspace optimization subproblems. Our preliminary results indicate that we can still define an analogue of Algorithm 2 and prove an analogue of Theorem 1, but the global convergence (an analogue of Theorem 2) seems more challenging.

5. ACKNOWLEDGEMENT

This work was supported by the ERC project "DECODA" no.320594, in the frame of the European program FP7/2007-2013. Lianze Li was partially supported by the National Natural Science Foundation of China (No.11601371).

6. REFERENCES

[1] P. Comon and C. Jutten, eds., *Handbook of Blind Source Separation*. Oxford: Academic Press, 2010.

[2] P. Comon, "Independent component analysis, a new concept ?," *Signal Processing*, vol. 36, no. 3, pp. 287–314, 1994.

[3] P. Comon, "Tensor Diagonalization, A useful Tool in Signal Processing," in *10th IFAC Symposium on System Identification (IFAC-SYSID)* (M. Blanke and T. Soderstrom, eds.), vol. 1, (Copenhagen, Denmark), pp. 77–82, IEEE, July 1994.

[4] J. Cardoso and A. Souloumiac, "Blind beamforming for non-gaussian signals," *IEE Proceedings F (Radar and Signal Processing)*, vol. 6, no. 140, pp. 362–370, 1993.

[5] L. De Lathauwer, B. De Moor, and J. Vandewalle, "Independent component analysis and (simultaneous) third-order tensor diagonalization," *IEEE Transactions on Signal Processing*, vol. 49, no. 10, pp. 2262–2271, 2001.

[6] J. Li, K. Usevich, and P. Comon, "Globally convergent Jacobi-type algorithms for simultaneous orthogonal symmetric tensor diagonalization," *SIAM Journal on Matrix Analysis and Applications*, vol. 39, no. 1, pp. 1–22, 2018.

[7] M. Ishteva, P.-A. Absil, and P. Van Dooren, "Jacobi algorithm for the best low multilinear rank approximation of symmetric tensors," *SIAM J. Matrix Anal. Appl.*, vol. 2, no. 34, pp. 651–672, 2013.